Hausdorff dimension and measure. Mass distribution principle.

What was wrong with Mink. dimension? Try to make a measure from it.
Cover K by sets of the same diameter ε. (Or at most ε).
$M_\alpha(K) = \inf \sum_{i=1}^{n} \varepsilon^\alpha = \inf N(\varepsilon,K)\varepsilon^\alpha$ — Minkowski content. Then $M_\alpha(K) = \begin{cases} 0, & \alpha > \overline{\mathrm{Mdim}}\, K \\ >0, & \alpha < \underline{\mathrm{Mdim}}\, K \end{cases}$ and we can describe $\underline{\mathrm{Mdim}}\, K$ as
$\underline{\mathrm{Mdim}}\, K = \sup\{\alpha : \overline{M_\alpha}(K)=0\} = \inf\{\alpha : M_\alpha(K)>0\}$

But, as we know, taking the sets of the same diameter does not work! For Lebesgue measure in d dimension we took $m_d(K) = \inf\{\sum \varepsilon_i^d : K \subset \cup K_i,\ \operatorname{diam} K_i = \varepsilon_i\}$. So, let us do the same for an arbitrary α. α-dimensional Hausdorff content is defined as

$$H_\alpha(K) := \inf\{\sum \varepsilon_i^\alpha : K \subset \cup K_i,\ \operatorname{diam} K_i = \varepsilon_i\}$$

Gauge function

Can even generalize it slightly. Let $h(t) \ge 0$ be a strictly increasing continuous function on $\mathbb{R}_+$, $h(0)=0$. Define h Hausdorff content as

$$H_h(K) := \inf\{\sum h(\varepsilon_i) : K \subset \cup K_i,\ \operatorname{diam} K_i = \varepsilon_i\}.$$

Same as what we had before for $h(t)=t^\alpha$.

Note that if K is countable, $H_h(K)=0$ (because for any ε we can cover $a_i \in K$ by disks of diameter ε_i with $h(\varepsilon_i) < 2^{-i}\varepsilon$.

Lemma 1. If $H_h(K)=0$ and $\lim_{t\to 0} \frac{g(t)}{h(t)} < \infty$, then $H_g(K)=0$.

Proof $\forall \varepsilon>0\ \exists$ covering K_j of K such that $\sum h(\operatorname{diam} K_j) < \varepsilon$.
Then, since h is strictly increasing, $\max \operatorname{diam} K_j \to 0$ as $\varepsilon \to 0$. Thus for some C, $g(\operatorname{diam} K_j) < C h(\operatorname{diam} K_j)$, so $\sum g(\operatorname{diam} K_j) < C\varepsilon \Rightarrow H_g(K) < C\varepsilon$ ■

Corollary 2. If $H_\alpha(K)=0$ and $\beta > \alpha$ then $H_\beta(K)=0$.
If $H_\alpha(K)>0$ and $\beta<\alpha$, then $H_\beta(K)>0$.

Similarly to the discussion of (lower) Minkowski dimension, can now define Hausdorff dimension as
$\operatorname{Hdim} K := \inf\{\alpha : H_\alpha(K)=0\} = \sup\{\alpha : H_\alpha(K)>0\}$

Of course, $\operatorname{Hdim} K \le \underline{\mathrm{Mdim}}\, K \le \overline{\mathrm{Mdim}}\, K$ (For any $\alpha > \underline{\mathrm{Mdim}}\, K$, cover by $N(\varepsilon_j, K)$ balls of radius ε_j, get $N(\varepsilon_j,K)\varepsilon_j^\alpha \to 0$).

One problem with H_h — it is not a measure.
Example. $H_{1/2}([0,1]) = H_{1/2}([1,2]) = 1$, but $H_{1/2}([0,2]) = \sqrt{2} < H_{1/2}([0,1]) + H_{1/2}([1,2])$. (simply because $a^{1/2}+b^{1/2} > (a+b)^{1/2}$)

To make it into a measure, force covering by smaller and smaller sets:

$$m_h^\varepsilon(K) := \inf\{\sum h(\varepsilon_j) : K \subset \cup K_j,\ \operatorname{diam} K_j = \varepsilon_j < \varepsilon\}$$

and
$m_h(K) := \lim_{\varepsilon \to 0} m_h^\varepsilon(K)$. The limit always exists (as a limit of an increasing function), but can be infinite.

m_h satisfies the following properties.

1) Monotonicity:
$K_1 \subset K_2 \Rightarrow m_h(K_1) \le m_h(K_2)$ — obvious, any cover of K_1 is a cover of K_2.

2) Subadditivity (countable)
$m_h(\bigcup_{j=1}^\infty K_j) \le \sum_{j=1}^\infty m_h(K_j)$ Pf Select cover of K_j with $\varepsilon_i < \varepsilon$ with $\sum h(\varepsilon_i) < m_h^\varepsilon(K_j) + \varepsilon 2^{-j}$, union of the covers will make the cover of $\cup K_j$ with $\sum h(\varepsilon_i) < \sum m_h^\varepsilon(K_j) + \varepsilon$. Let $\varepsilon \to 0$. ■

3) Metric separated additivity:
if $\operatorname{dist}(K_1,K_2)>0$ then $m_h(K_1)+m_h(K_2) = m_h(K_1 \cup K_2)$.
Pf. When $\varepsilon < \operatorname{dist}(K_1,K_2)$, the covers of K_1 and K_2 do not know about each other ■

Def. A set function satisfying 1)-3) is called Metric Outer measure.

Thm. Any metric outer measure restricted to Borel sets is a measure.
No proof.

Property $m_h(K) \ge H_h(K)$ and $H_h(K)=0 \Leftrightarrow m_h(K)=0$

Proof The first statement follows from the definition.
[illegible]

Second notice that if K_i is a covering such that $\sum h(\operatorname{diam} K_i) < \varepsilon$ then $\max \operatorname{diam} K_i < h^{-1}(\varepsilon) \to 0$ as $\varepsilon \to 0$.
so $m_{h, h^{-1}(\varepsilon)}(K) < \varepsilon$. Let $\varepsilon \to 0$. ∎

Lemma. If h, g are two gauge functions, $\lim_{t \to 0} \frac{h(t)}{g(t)} = 0$, and $m_g(K) < \infty$ then $m_h(K) = 0$

Pf. Fix any $\delta > 0$ and choose $\varepsilon > 0$ such that $t < \varepsilon \Rightarrow h(t) < \delta g(t)$.
Consider a covering of K by K_i such that $\operatorname{diam} K_i < \varepsilon$ and $\sum g(K_i) < m_{g,\varepsilon}(K) + \varepsilon$. Then
$m_{h,\varepsilon}(K) \le \sum h(K_i) < \delta \sum g(K_i) < \delta (m_{g,\varepsilon}(K) + \varepsilon)$.
Now let $\varepsilon \to 0$ to see that $m_h(K) < \delta$. ∎

Corollaries:

1) If $\lim_{t \to 0} \frac{h(t)}{g(t)} = 0$, $m_h(K) > 0$, then $m_g(K) = \infty$

2) If $\alpha = \operatorname{Hdim} K$ and $\beta < \alpha < \gamma$, then $m_\beta(K) = \infty$ and $m_\gamma(K) = 0$

Thus $\operatorname{Hdim}(K) = \inf\{\alpha : m_\alpha(K) = 0\} = \sup\{\alpha : m_\alpha(K) = \infty\}$.

So we can estimate the Hausdorff dimension from above, by presenting a cover. How to estimate it bellow?

Def. A measure μ is called *h-smooth* if for some C and for every ball $B(x,r)$, $\mu(B(x,r)) \le Ch(r)$.

Thm (Mass distribution principle). Let $\mu(K) > 0$ for some h-smooth measure, then $m_h(K) \ge H_h(K) > \frac{\mu(K)}{C}$, where C is the constant in the definition of h-smoothness

Proof. Let (K_i) be any cover of K. Then $K_i \subset B(x_i, \operatorname{diam} K_i)$ for any $x_i \in K_i$.
Then $\mu(K) \le \sum \mu(K_i) \le \sum C h(\operatorname{diam} K_i)$. Take inf over all the coverings. ∎

Corollary. If $\mu(K) > 0$ for some α-smooth measure ($\mu(B(x,r)) \le Cr^\alpha$) then $\operatorname{Hdim} K \ge \alpha$.

Using this, it is easy to prove that $\operatorname{Hdim} C = \frac{\log 2}{\log 3}$ (C – the usual Cantor set).
Construct μ by assigning $\mu(I_n^k) = 2^{-n}$ for any interval $I_n^k \in C_n$. $\mu(C) = 1$, and notice that for $3^{-n} \le r < 3^{-n+1}$, $B(x,r)$ intersects at most one I_{n-1}^k, so $\mu(B(x,r)) \le 2^{-n+1} \le 2r^{\frac{\log 2}{\log 3}}$, so μ is $\frac{\log 2}{\log 3}$-smooth.
Thus $\frac{\log 2}{\log 3} \le \operatorname{Hdim} C \le \operatorname{Mdim} C = \frac{\log 2}{\log 3}$.

For our Cantor set example (defined by a sequence (ℓ_n))
Let us observe that
$H_h(C) > 0 \Leftrightarrow \underline{\lim}_{n \to \infty} 2^{nd} h(\ell_n) > 0$

Proof.
$\Rightarrow$ is by the fact that for k with $2^k > \sqrt{d}$, $2^{n(d+k)}$ cubes of diameter $\sqrt{d}\,\ell_{n+k} < \ell_n$ cover C.
$\Leftarrow$ is because the usual measure giving equal mass 2^{-nd} to each cube of generation n is h-smooth.
Indeed, for $\ell_n \le \varepsilon < \ell_{n-1}$, $B(x,\varepsilon)$ intersects at most 2^{2d} cubes of generation n. Thus $\mu(B(x,\varepsilon)) \le 2^{2d} 2^{-nd} \le 2^{2d} C h(\ell_n) \le C 2^{2d} h(\varepsilon)$,
where $C^{-1} := \underline{\inf}\, 2^{nd} h(\ell_n) > 0$

In particular, $\operatorname{Hdim} C = \underline{\operatorname{Mdim}}\, C = \frac{1}{\overline{\lim} \log_2 \sqrt[nd]{\ell_n}}$.